Problem 1. *Show that if* a, b, c *are positive real numbers, then*

$$\frac{a^4}{b^2}+\frac{b^4}{c^2}+\frac{c^4}{a^2}+5(ab+bc+ca)\ge 6(a^2+b^2+c^2).$$

Proof. Without loss of generality, assume that $c=\min\{a,b,c\}.$ There are two cases:

1. *Case* $a\ge b\ge c.$ Write the inequality as

$$\left(\frac{a^4}{b^2}+\frac{b^4}{c^2}+\frac{c^4}{a^2}-a^2-b^2-c^2\right)\ge 5(a^2+b^2+c^2-ab-bc-ca).$$

For all $x, y, z>0,$ we have

$$\begin{aligned}\frac{x^4}{y^2}+\frac{y^4}{z^2}+\frac{z^4}{x^2}-x^2-y^2-z^2&=\left(\frac{x^4}{y^2}+\frac{y^4}{x^2}-x^2-y^2\right)+\left(\frac{y^4}{z^2}+\frac{z^4}{x^2}-\frac{y^4}{x^2}-z^2\right)\\&=\frac{(x^2-y^2)^2(x^2+y^2)}{x^2y^2}+\frac{(x^2-z^2)(y^2-z^2)(y^2+z^2)}{x^2z^2}\end{aligned}$$

and

$$\begin{aligned}x^2+y^2+z^2-xy-yz-zx&=(x^2+y^2-2xy)+(z^2+xy-yz-zx)\\&=(x-y)^2+(x-z)(y-z).\end{aligned}$$

Therefore, the above inequality can be written in the following forms (choosing $(x,y,z)=(a,b,c)$ and $(x,y,z)=(b,c,a),$ respectively)

$$(a-b)^2\left[\frac{(a+b)^2(a^2+b^2)}{a^2b^2}-5\right]+(a-c)(b-c)\left[\frac{(b^2+c^2)(a+c)(b+c)}{a^2c^2}-5\right]\ge 0 \quad (1)$$

and

$$(b-c)^2\left[\frac{(b+c)^2(b^2+c^2)}{b^2c^2}-5\right]+(a-b)(a-c)\left[\frac{(a^2+c^2)(a+b)(a+c)}{a^2b^2}-5\right]\ge 0. \quad (2)$$

- *If* $b\ge \frac{a+c}{2},$ then we have

$$\begin{aligned}\frac{(b^2+c^2)(a+c)(b+c)}{a^2c^2}&\ge\frac{\left[\frac{(a+c)^2}{4}+c^2\right](a+c)\left(\frac{a+c}{2}+c\right)}{a^2c^2}\\&\ge\frac{(ac+c^2)(a+c)(a+3c)}{2a^2c^2}=\frac{(a+c)^2(a+3c)}{2a^2c}\\&=\frac{\left[2a+(a+3c)\right]^2(a+3c)}{18a^2c}\ge\frac{\left[4\cdot 2a\cdot(a+3c)\right](a+3c)}{18a^2c}\\&=\frac{4(a+3c)^2}{9ac}\ge\frac{4(4\cdot a\cdot 3c)}{9ac}=\frac{16}{3}>5.\end{aligned}$$

  Combining this with the obvious inequality $(a-c)(b-c)\ge 0,$ we get

$$(a-c)(b-c)\left[\frac{(b^2+c^2)(a+c)(b+c)}{a^2c^2}-5\right]\ge 0 \quad (3)$$

  On the other hand, it is clear that

$$(a-b)^2\left[\frac{(a+b)^2(a^2+b^2)}{a^2b^2}-5\right]\ge 0. \quad (4)$$

From (3) and (4), we can see that (1) holds and so the original inequality is proved in this case.

- *If* $a+c\ge 2b,$ we will consider two cases.
  - *Case* $a\ge 2b.$ In this case, we have
$$\frac{(a^2+c^2)(a+b)(a+c)}{a^2b^2}\ge\frac{a^2\cdot(a+b)\cdot a}{a^2b^2}=\frac{a(a+b)}{b^2}\ge\frac{2b(2b+b)}{b^2}=6>5,$$
and since $(a-b)(a-c)\ge 0,$ it follows that
$$(a-b)(a-c)\left[\frac{(a^2+c^2)(a+b)(a+c)}{a^2b^2}-5\right]\ge 0. \qquad (5)$$
Moreover, one can easily check that
$$(b-c)^2\left[\frac{(b+c)^2(b^2+c^2)}{b^2c^2}-5\right]\ge 0. \qquad (6)$$
So (2) holds or the original inequality is proved.
  - *Case* $2b\ge a.$ Since $a+c\ge 2b,$ we have $c\ge 2b-a\ge 0$ and hence
$$\begin{aligned}\frac{(a^2+c^2)(a+b)(a+c)}{a^2b^2}&\ge\frac{\left[a^2+(2b-a)^2\right](a+b)(2b)}{a^2b^2}\\&=\frac{4(a+b)(a^2-2ab+2b^2)}{a^2b}=\frac{4(a^3+2b^3-a^2b)}{a^2b}\\&=\frac{4\left[\left(\frac{a^3}{2}+\frac{a^3}{2}+2b^3\right)-a^2b\right]}{a^2b}\ge\frac{4\left(\frac{3}{\sqrt[3]{2}}a^2b-a^2b\right)}{a^2b}\\&=4\left(\frac{3}{\sqrt[3]{2}}-1\right)>5.\end{aligned}$$
From this and the obvious inequality $(a-b)(a-c)\ge 0,$ we also get (5). And by combining (5) and the obvious inequality (6), we obtain the desired result.

2. *Case* $b\ge a\ge c.$ In this case, we have
$$\frac{a^4}{b^2}+\frac{b^4}{c^2}+\frac{c^4}{a^2}\ge\frac{b^4}{a^2}+\frac{c^4}{b^2}+\frac{a^4}{c^4}$$ (the readers can easily check this).
Therefore, it suffices to prove that
$$\frac{b^4}{a^2}+\frac{c^4}{b^2}+\frac{a^4}{c^4}+5(ab+bc+ca)\ge 6(a^2+b^2+c^2).$$
Setting $a'=b, b'=a$ and $c'=c,$ the above inequality becomes
$$\frac{a'^4}{b'^2}+\frac{b'^4}{c'^2}+\frac{c'^4}{a'^2}+5(a'b'+b'c'+c'a')\ge 6(a'^2+b'^2+c'^2).$$
The "new" inequality has exactly the same form with the original inequality and in this inequality, we have $a'\ge b'\ge c'.$ According to the result of the first case, this inequality is true. So our proof is completed. □